Physics 513, Quantum Field Theory
Examination 1

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Jacob Lewis Bourjaily
University of Michigan, Department of Physics, Ann Arbor, MI 48109-1120

1. a) We are to verify that in the Schödinger picture we may write the total momentum operator,

$$\mathbf{P} = -\int d^3x\ \pi(\mathbf{x})\nabla\phi(\mathbf{x}),$$

in terms of ladder operators as

$$\mathbf{P} = \int \frac{d^3p}{(2\pi)^3}\ \mathbf{p}\ a^\dagger_{\mathbf{p}} a_{\mathbf{p}}.$$

Recall that in the Schrödinger picture, we have the following expansions for the fields ϕ and π in terms of the bosonic ladder operators

$$\phi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{\sqrt{2E_{\mathbf{p}}}} e^{i\mathbf{px}} \left(a_{\mathbf{p}} + a^\dagger_{-\mathbf{p}}\right); \tag{1.1}$$

$$\pi(\mathbf{x}) = \int \frac{d^3p}{(2\pi)^3}\ (-i)\sqrt{\frac{E_{\mathbf{p}}}{2}} e^{i\mathbf{px}} \left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right). \tag{1.2}$$

To begin our derivation, let us compute $\vec{\nabla}\phi(\mathbf{x})$.

$$\begin{aligned}
\nabla\phi(\mathbf{x}) &= \nabla \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(a_{\mathbf{p}} e^{i\mathbf{px}} + a^\dagger_{\mathbf{p}} e^{-i\mathbf{px}}\right), \\
&= \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{\sqrt{2E_{\mathbf{p}}}} \left(i\mathbf{p} a_{\mathbf{p}} e^{i\mathbf{px}} - i\mathbf{p} a^\dagger_{\mathbf{p}} e^{-i\mathbf{px}}\right), \\
&= \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{\sqrt{2E_{\mathbf{p}}}} i\mathbf{p} e^{i\mathbf{px}} \left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right).
\end{aligned}$$

Using this and (1.2) we may write the expression for $\mathbf{P}$ directly.

$$\begin{aligned}
\mathbf{P} &= -\int d^3x\ \pi(\mathbf{x})\nabla\phi(\mathbf{x}), \\
&= -\int d^3x \frac{d^3k d^3p}{(2\pi)^6}\ \frac{1}{2}\sqrt{\frac{E_{\mathbf{k}}}{E_{\mathbf{p}}}} \mathbf{p} e^{i(\mathbf{p}+\mathbf{k})\mathbf{x}} \left(a_{\mathbf{k}} - a^\dagger_{-\mathbf{k}}\right)\left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right), \\
&= \int \frac{d^3k d^3p}{(2\pi)^6}\ \frac{-1}{2}\sqrt{\frac{E_{\mathbf{k}}}{E_{\mathbf{p}}}} \mathbf{p}(2\pi)^3\delta^{(3)}(\mathbf{p}+\mathbf{k}) \left(a_{\mathbf{k}} - a^\dagger_{-\mathbf{k}}\right)\left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right), \\
&= \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{2}\mathbf{p}\left(a^\dagger_{\mathbf{p}} - a_{-\mathbf{p}}\right)\left(a_{\mathbf{p}} - a^\dagger_{-\mathbf{p}}\right).
\end{aligned}$$

Using symmetry we may show that $a_{-\mathbf{p}} a^\dagger_{-\mathbf{p}} = a_{\mathbf{p}} a^\dagger_{\mathbf{p}}$. With this, our total momentum becomes,

$$\mathbf{P} = \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{2}\mathbf{p}\left(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + a_{\mathbf{p}} a^\dagger_{\mathbf{p}}\right).$$

By adding and then subtracting $a^\dagger_{\mathbf{p}} a_{\mathbf{p}}$ inside the parenthesis, one sees that

$$\begin{aligned}
\mathbf{P} &= \int \frac{d^3p}{(2\pi)^3}\ \frac{1}{2}\mathbf{p}\left(2a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + [a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}]\right), \\
&= \int \frac{d^3p}{(2\pi)^3} \mathbf{p}\left(a^\dagger_{\mathbf{p}} a_{\mathbf{p}} + \tfrac{1}{2}[a_{\mathbf{p}}, a^\dagger_{\mathbf{p}}]\right).
\end{aligned}$$

Unfortunately, we have precisely the same problem that we had with the Hamiltonian: there is an infinite 'baseline' momentum. Of course, our 'justification' here will be identical to the one offered in that case and so

$$\boxed{\therefore \mathbf{P} = \int \frac{d^3p}{(2\pi)^3}\ \mathbf{p}\ a^\dagger_{\mathbf{p}} a_{\mathbf{p}}.} \tag{1.3}$$

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b) We are to verify that the Dirac charge operator,

$$Q = \int d^3x \ \psi^\dagger(x)\psi(x),$$

may be written in terms of ladder operators as

$$Q = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s\right).$$

Recall that we can expand our Dirac ψ's in terms of fermionic ladder operators.

$$\psi_a(x) = \int \frac{d^3p}{(2\pi)^3} \ \frac{1}{\sqrt{2E_{\mathbf{p}}}} e^{i\mathbf{px}} \sum_s \Big(a_{\mathbf{p}}^s u_a^s(\mathbf{p}) + b_{-\mathbf{p}}^{s\dagger} v_a^s(-\mathbf{p})\Big); \tag{1.4}$$

$$\psi_b(x)^\dagger = \int \frac{d^3p}{(2\pi)^3} \ \frac{1}{\sqrt{2E_{\mathbf{p}}}} e^{-i\mathbf{px}} \sum_r \Big(a_{\mathbf{p}}^{r\dagger} u_b^{r\dagger}(\mathbf{p}) + b_{-\mathbf{p}}^r v_b^{r\dagger}(-\mathbf{p})\Big). \tag{1.5}$$

Therefore, we can compute Q by writing out its terms explicitly.

$$\begin{aligned}
Q &= \int d^3x \ \psi^\dagger(x)\psi(x), \\
&= \int d^3x \frac{d^3k d^3p}{(2\pi)^6} \frac{1}{2\sqrt{E_{\mathbf{k}} E_{\mathbf{p}}}} e^{i(\mathbf{p}-\mathbf{k})\mathbf{x}} \sum_{r,s} \Big[\Big(a_{\mathbf{k}}^{r\dagger} u_b^{r\dagger}(\mathbf{k}) + b_{-\mathbf{k}}^r v_b^{r\dagger}(-\mathbf{k})\Big)\Big(a_{\mathbf{p}}^s u_a^s(\mathbf{p}) + b_{-\mathbf{p}}^{s\dagger} v_a^s(-\mathbf{p})\Big)\Big], \\
&= \int \frac{d^3k d^3p}{(2\pi)^6} \frac{1}{2\sqrt{E_{\mathbf{k}} E_{\mathbf{p}}}} (2\pi)^3 \delta^{(3)}(\mathbf{p}-\mathbf{k}) \sum_{r,s} \Big[\Big(a_{\mathbf{k}}^{r\dagger} u_b^{r\dagger}(\mathbf{k}) + b_{-\mathbf{k}}^r v_b^{r\dagger}(-\mathbf{k})\Big)\Big(a_{\mathbf{p}}^s u_a^s(\mathbf{p}) + b_{-\mathbf{p}}^{s\dagger} v_a^s(-\mathbf{p})\Big)\Big], \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_{r,s} \Big[\Big(a_{\mathbf{p}}^{r\dagger} u_b^{r\dagger}(\mathbf{p}) + b_{-\mathbf{p}}^r v_b^{r\dagger}(-\mathbf{p})\Big)\Big(a_{\mathbf{p}}^s u_a^s(\mathbf{p}) + b_{-\mathbf{p}}^{s\dagger} v_a^s(-\mathbf{p})\Big)\Big], \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_{r,s} \Big(a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^s u_b^{r\dagger}(\mathbf{p}) u_a^s(\mathbf{p}) + a_{\mathbf{p}}^{r\dagger} b_{-\mathbf{p}}^{s\dagger} u_b^{r\dagger}(\mathbf{p}) v_a^s(-\mathbf{p}) \\
&\qquad + b_{-\mathbf{p}}^r a_{\mathbf{p}}^s v_b^{r\dagger}(-\mathbf{p}) u_a^s(\mathbf{p}) + b_{-\mathbf{p}}^r b_{-\mathbf{p}}^{s\dagger} v_b^{r\dagger}(-\mathbf{p}) v_a^s(-\mathbf{p})\Big), \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} \sum_{r,s} \Big(a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^s u_b^{r\dagger}(\mathbf{p}) u_a^s(\mathbf{p}) + b_{-\mathbf{p}}^r b_{-\mathbf{p}}^{s\dagger} v_b^{r\dagger}(-\mathbf{p}) v_a^s(-\mathbf{p})\Big), \\
&= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\mathbf{p}}} 2E_{\mathbf{p}} \delta^{rs} \sum_{r,s} \Big(a_{\mathbf{p}}^{r\dagger} a_{\mathbf{p}}^s + b_{-\mathbf{p}}^r b_{-\mathbf{p}}^{s\dagger}\Big), \\
&= \int \frac{d^3p}{(2\pi)^3} \sum_s \Big(a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s + b_{-\mathbf{p}}^s b_{-\mathbf{p}}^{s\dagger}\Big),
\end{aligned}$$

We note that by symmetry $b_{-\mathbf{p}}^s b_{-\mathbf{p}}^{s\dagger} = b_{\mathbf{p}}^s b_{\mathbf{p}}^{s\dagger}$. By using its anticommutation relation to rewrite $b_{\mathbf{p}}^s b_{\mathbf{p}}^{s\dagger}$ and then dropping the infinite 'baseline' energy as we did in part (a), we see that

$$\boxed{\therefore Q = \int \frac{d^3p}{(2\pi)^3} \sum_s \left(a_{\mathbf{p}}^{s\dagger} a_{\mathbf{p}}^s - b_{\mathbf{p}}^{s\dagger} b_{\mathbf{p}}^s\right).} \tag{1.6}$$

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2. a) We are to show that the matrices

$$(\mathcal{J}^{\mu\nu})_{\alpha\beta} = i\left(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}\right),$$

generate the Lorentz algebra,

$$[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i\left(g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\mathcal{J}^{\nu\sigma} - g^{\nu\sigma}\mathcal{J}^{\mu\rho} + g^{\mu\sigma}\mathcal{J}^{\nu\rho}\right).$$

We are reminded that matrix multiplication is given by $(AB)_{\alpha\gamma} = A_{\alpha\beta}B^{\beta}{}_{\gamma}$. Recall that in homework 5.1, we showed that

$$(\mathcal{J}^{\mu\nu})^{\alpha}{}_{\beta} = i\left(g^{\mu\alpha}\delta^{\nu}{}_{\beta} - g^{\nu\alpha}\delta^{\mu}{}_{\beta}\right).$$

Let us proceed directly to demonstrate the Lorentz algebra.

$$\begin{aligned}
[\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] &= -\left(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta} - \delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}\right)\left(g^{\rho\beta}\delta^{\sigma}{}_{\gamma} - g^{\sigma\beta}\delta^{\rho}{}_{\gamma}\right) + \left(\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta} - \delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\alpha}\right)\left(g^{\mu\beta}\delta^{\nu}{}_{\gamma} - g^{\nu\beta}\delta^{\mu}{}_{\gamma}\right),\\
&= -\underbrace{\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}g^{\rho\beta}\delta^{\sigma}{}_{\gamma}}_{1} + \underbrace{\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}g^{\sigma\beta}\delta^{\rho}{}_{\gamma}}_{2} + \underbrace{\delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}g^{\rho\beta}\delta^{\sigma}{}_{\gamma}}_{3} - \underbrace{\delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}g^{\sigma\beta}\delta^{\rho}{}_{\gamma}}_{4}\\
&\qquad + \underbrace{\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}g^{\mu\beta}\delta^{\nu}{}_{\gamma}}_{5} - \underbrace{\delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}g^{\nu\beta}\delta^{\mu}{}_{\gamma}}_{6} - \underbrace{\delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\alpha}g^{\mu\beta}\delta^{\nu}{}_{\gamma}}_{7} + \underbrace{\delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\alpha}g^{\nu\beta}\delta^{\mu}{}_{\gamma}}_{8},\\
&= -\underbrace{(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}g^{\rho\beta}\delta^{\sigma}{}_{\gamma} - \delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\alpha}g^{\nu\beta}\delta^{\mu}{}_{\gamma})}_{1\&8}g^{\nu\rho} + \underbrace{(\delta^{\mu}{}_{\alpha}\delta^{\nu}{}_{\beta}g^{\sigma\beta}\delta^{\rho}{}_{\gamma} - \delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}g^{\nu\beta}\delta^{\mu}{}_{\gamma})}_{2\&6}g^{\nu\sigma}\\
&\qquad + \underbrace{(\delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}g^{\rho\beta}\delta^{\sigma}{}_{\gamma} - \delta^{\rho}{}_{\beta}\delta^{\sigma}{}_{\alpha}g^{\mu\beta}\delta^{\nu}{}_{\gamma})}_{3\&7}g^{\mu\rho} - \underbrace{(\delta^{\mu}{}_{\beta}\delta^{\nu}{}_{\alpha}g^{\sigma\beta}\delta^{\rho}{}_{\gamma} - \delta^{\rho}{}_{\alpha}\delta^{\sigma}{}_{\beta}g^{\mu\beta}\delta^{\nu}{}_{\gamma})}_{4\&5}g^{\mu\sigma},
\end{aligned}$$

$$\boxed{\therefore [\mathcal{J}^{\mu\nu}, \mathcal{J}^{\rho\sigma}] = i\left(g^{\nu\rho}\mathcal{J}^{\mu\sigma} - g^{\mu\rho}\mathcal{J}^{\nu\sigma} - g^{\nu\sigma}\mathcal{J}^{\mu\rho} + g^{\mu\sigma}\mathcal{J}^{\nu\rho}\right).} \tag{2.1}$$

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b) Like part (a) above, we are to show that the matrices

$$S^{\mu\nu} = \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}],$$

generate the Lorentz algebra,

$$[S^{\mu\nu}, S^{\rho\sigma}] = i\left(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}\right).$$

As Pascal wrote, 'I apologize for the length of this [proof], for I did not have time to make it short.' Before we proceed directly, let's outline the derivation so that the algebra is clear. First, we will fully expand the commutator of $S^{\mu\nu}$ with $S^{\rho\sigma}$. We will have 8 terms. For each of those terms, we will use the anticommutation identity $\gamma^{\mu}\gamma^{\nu} = 2g^{\mu\nu} - \gamma^{\nu}\gamma^{\mu}$ to rewrite the middle of each term. By repeated use of the anticommutation relations, it can be shown that

$$\gamma^{\mu}\gamma^{\rho}\gamma^{\nu}\gamma^{\sigma} = \gamma^{\sigma}\gamma^{\nu}\gamma^{\rho}\gamma^{\mu} + 2(g^{\nu\sigma}\gamma^{\mu}\gamma^{\rho} - g^{\rho\sigma}\gamma^{\mu}\gamma^{\nu} + g^{\mu\sigma}\gamma^{\rho}\gamma^{\nu} - g^{\rho\nu}\gamma^{\sigma}\gamma^{\mu} + g^{\mu\nu}\gamma^{\sigma}\gamma^{\rho} - g^{\mu\rho}\gamma^{\sigma}\gamma^{\nu}), \tag{2.2}$$

This will be used to cancel many terms and multiply the whole expression by 2 before we contract back to terms involving $S^{\mu\nu}$'s. Let us begin.

$$\begin{aligned}
[S^{\mu\nu}, S^{\rho\sigma}] &= -\frac{1}{16}\left([\gamma^{\mu}\gamma^{\nu} - \gamma^{\nu}\gamma^{\mu}, \gamma^{\rho}\gamma^{\sigma} - \gamma^{\sigma}\gamma^{\rho}]\right),\\
&= -\frac{1}{16}\left([\gamma^{\mu}\gamma^{\nu}, \gamma^{\rho}\gamma^{\sigma}] - [\gamma^{\mu}\gamma^{\nu}, \gamma^{\sigma}\gamma^{\rho}] - [\gamma^{\nu}\gamma^{\mu}, \gamma^{\rho}\gamma^{\sigma}] + [\gamma^{\nu}\gamma^{\mu}, \gamma^{\sigma}\gamma^{\rho}]\right),\\
&= -\frac{1}{16}\left(\gamma^{\mu}\gamma^{\nu}\gamma^{\rho}\gamma^{\sigma} - \gamma^{\rho}\gamma^{\sigma}\gamma^{\mu}\gamma^{\nu} - \gamma^{\mu}\gamma^{\nu}\gamma^{\sigma}\gamma^{\rho} + \gamma^{\sigma}\gamma^{\rho}\gamma^{\mu}\gamma^{\nu}\right.\\
&\qquad\qquad \left.-\gamma^{\nu}\gamma^{\mu}\gamma^{\rho}\gamma^{\sigma} + \gamma^{\rho}\gamma^{\sigma}\gamma^{\nu}\gamma^{\mu} + \gamma^{\nu}\gamma^{\mu}\gamma^{\sigma}\gamma^{\rho} - \gamma^{\sigma}\gamma^{\rho}\gamma^{\nu}\gamma^{\mu}\right),\\
&= -\frac{1}{16}\Big(2g^{\nu\rho}\gamma^{\mu}\gamma^{\sigma} - \gamma^{\mu}\gamma^{\rho}\gamma^{\nu}\gamma^{\sigma} - 2g^{\nu\rho}\gamma^{\sigma}\gamma^{\mu} + \gamma^{\sigma}\gamma^{\nu}\gamma^{\rho}\gamma^{\mu}\\
&\qquad - 2g^{\sigma\mu}\gamma^{\rho}\gamma^{\nu} + \gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}\gamma^{\nu} + 2g^{\sigma\mu}\gamma^{\nu}\gamma^{\rho} - \gamma^{\nu}\gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}\\
&\qquad - 2g^{\nu\sigma}\gamma^{\mu}\gamma^{\rho} + \gamma^{\mu}\gamma^{\sigma}\gamma^{\nu}\gamma^{\rho} + 2g^{\nu\sigma}\gamma^{\rho}\gamma^{\mu} - \gamma^{\rho}\gamma^{\nu}\gamma^{\sigma}\gamma^{\mu}\\
&\qquad + 2g^{\mu\rho}\gamma^{\sigma}\gamma^{\nu} - \gamma^{\sigma}\gamma^{\mu}\gamma^{\rho}\gamma^{\nu} - 2g^{\mu\rho}\gamma^{\nu}\gamma^{\sigma} + \gamma^{\nu}\gamma^{\rho}\gamma^{\mu}\gamma^{\sigma}\Big).
\end{aligned}$$

Now, the rest of the derivation is a consequence of (2.2). Because each $\gamma^\mu\gamma^\nu\gamma^\rho\gamma^\sigma$ term is equal to its complete antisymmetrization $\gamma^\sigma\gamma^\rho\gamma^\nu\gamma^\mu$ together with six $g^{\nu\sigma}$-like terms, all terms not involving the metric tensor will cancel each other. When we add all of the contributions from all of the cancellings, sixteen of the added twenty-four terms will cancel each other and the eight remaining will have the effect of multiplying each of the $g^{\nu\sigma}$-like terms by two. So after this is done in a couple of pages of algebra that I am not courageous enough to type, the commutator is reduced to

$$[S^{\mu\nu}, S^{\rho\sigma}] = \frac{1}{4}\left(-g^{\nu\rho}(\gamma^\mu\gamma^\sigma - \gamma^\sigma\gamma^\mu) - g^{\mu\sigma}(\gamma^\nu\gamma^\rho - \gamma^\rho\gamma^\nu) + g^{\nu\sigma}(\gamma^\mu\gamma^\rho - \gamma^\rho\gamma^\mu) + g^{\mu\rho}(\gamma^\nu\gamma^\sigma - \gamma^\sigma\gamma^\nu)\right).$$

$$\boxed{\therefore [S^{\mu\nu}, S^{\rho\sigma}] = i\left(g^{\nu\rho}S^{\mu\sigma} - g^{\mu\rho}S^{\nu\sigma} - g^{\nu\sigma}S^{\mu\rho} + g^{\mu\sigma}S^{\nu\rho}\right).} \tag{2.3}$$

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c) We are to show the explicit formulations of the Lorentz boost matrices $\Lambda(\eta)$ along the x^3 direction in both vector and spinor representations. These are generically given by

$$\Lambda(\omega) = e^{-\frac{i}{2}\omega_{\mu\nu}J^{\mu\nu}},$$

where $J^{\mu\nu}$ are the representation matrices of the algebra and $\omega_{\mu\nu}$ parameterize the transformation group element.

In the vector representation, this matrix is,

$$\boxed{\Lambda(\eta) = \begin{pmatrix} \cosh(\eta) & 0 & 0 & \sinh(\eta) \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \sinh(\eta) & 0 & 0 & \cosh(\eta) \end{pmatrix}.} \tag{2.4}$$

In the spinor representation, this matrix is

$$\Lambda(\eta) = \begin{pmatrix} \cosh(\eta/2) - \sinh(\eta/2) & 0 & 0 & 0 \\ 0 & \cosh(\eta/2) + \sinh(\eta/2) & 0 & 0 \\ 0 & 0 & \cosh(\eta/2) + \sinh(\eta/2) & 0 \\ 0 & 0 & 0 & \cosh(\eta/2) - \sinh(\eta/2) \end{pmatrix}$$

So,

$$\boxed{\Lambda(\eta) = \begin{pmatrix} e^{-\eta/2} & 0 & 0 & 0 \\ 0 & e^{\eta/2} & 0 & 0 \\ 0 & 0 & e^{\eta/2} & 0 \\ 0 & 0 & 0 & e^{-\eta/2} \end{pmatrix}.} \tag{2.5}$$

d) No components of the Dirac spinor are invariant under a nontrivial boost.

e) Like part (c) above, we are to explicitly write out the rotation matrices $\Lambda(\theta)$ corresponding to a rotation about the x^3 axis.

In the vector representation, this matrix is given by

$$\boxed{\Lambda(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.} \tag{2.6}$$

In the spinor representation, this matrix is given by

$$\boxed{\Lambda(\theta) = \begin{pmatrix} e^{-i\theta/2} & 0 & 0 & 0 \\ 0 & e^{i\theta/2} & 0 & 0 \\ 0 & 0 & e^{-i\theta/2} & 0 \\ 0 & 0 & 0 & e^{i\theta/2} \end{pmatrix}.} \tag{2.7}$$

f) The vectors are symmetric under 2π rotations and so are unchanged under a 'complete' rotation. Spinors, however, are symmetric under 4π rotations are therefore only 'half-way back' under a 2π rotation.

3. a) Let us define the chiral transformation to be given by $\psi \to e^{i\alpha\gamma^5}\psi$. How does the conjugate spinor $\bar{\psi}$ transform?

We may begin to compute this transformation directly.

$$\begin{aligned}\bar{\psi} \to \bar{\psi}' &= \psi'^\dagger \gamma^0, \\ &= (e^{i\alpha\gamma^5}\psi)^\dagger \gamma^0, \\ &= \psi^\dagger e^{-i\alpha\gamma^5}\gamma^0.\end{aligned}$$

When we expand $e^{-i\alpha\gamma^5}$ in its Taylor series, we see that because γ^0 anticommutes with each of the γ^5 terms, we may bring the γ^0 to the left of the exponential with the cost of a change in the sign of the exponent. Therefore

$$\boxed{\bar{\psi} \to \bar{\psi} e^{i\alpha\gamma^5}.} \tag{3.1}$$

b) We are to show the transformation properties of the vector $V^\mu = \bar{\psi}\gamma^\mu\psi$.

We can compute this transformation directly. Note that γ^5 anticommutes with all γ^μ.

$$\begin{aligned}V^\mu = \bar{\psi}\gamma^\mu\psi \to &\ \bar{\psi}e^{i\alpha\gamma^5}\gamma^\mu e^{i\alpha\gamma^5}\psi, \\ &= \bar{\psi}\gamma^\mu e^{-i\alpha\gamma^5}e^{i\alpha\gamma^5}\psi, \\ &= \bar{\psi}\gamma^\mu\psi = V^\mu.\end{aligned}$$

Therefore,

$$\boxed{V^\mu \to V^\mu.} \tag{3.2}$$

c) We must show that the Dirac Lagrangian $\mathcal{L} = \bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$ is invariant under chiral transformations in the the massless case but is not so when $m \neq 0$.

Note that because the vectors are invariant, $\partial_\mu \to \partial_\mu$. Therefore, we may directly compute the transformation in each case. Let us say that $m = 0$.

$$\begin{aligned}\mathcal{L} = \bar{\psi}i\gamma^\mu\partial_\mu\psi \to \mathcal{L}' &= \bar{\psi}ie^{i\alpha\gamma^5}\gamma^\mu e^{i\alpha\gamma^5}\partial_\mu, \\ &= \bar{\psi}i\gamma^\mu e^{-i\alpha\gamma^5}e^{i\alpha\gamma^5}\partial_\mu\psi, \\ &= \bar{\psi}i\gamma^\mu\partial_\mu\psi = \mathcal{L}.\end{aligned}$$

Therefore the Lagrangian is invariant if $m = 0$. On the other hand, if $m \neq 0$,

$$\begin{aligned}\mathcal{L} = \bar{\psi}i\gamma^\mu\partial_\mu\psi - \bar{\psi}m\psi \to \mathcal{L}' &= \bar{\psi}i\gamma^\mu\partial_\mu\psi - \bar{\psi}e^{i\alpha\gamma^5}me^{i\alpha\gamma^5}\psi, \\ &= \bar{\psi}i\gamma^\mu\partial_\mu\psi - \bar{\psi}me^{2i\alpha\gamma^5}\psi \neq \mathcal{L}.\end{aligned}$$

It is clear that the Lagrangian is not invariant under the chiral transformation generally.

d) The most general Noether current is

$$j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\delta\phi(x) - \left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)}\partial_\nu\phi(x) - \mathcal{L}\delta^\mu{}_\nu\right)\delta x^\nu,$$

where $\delta\phi$ is the total variation of the field and δx^ν is the coordinate variation. In the chiral transformation, $\delta x^\nu = 0$ and ϕ is the Dirac spinor field. So the Noether current in our case is given by,

$$j_5^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)}\delta\psi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})}\delta\bar{\psi}.$$

Now, first we note that

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\psi)} = \bar{\psi}i\gamma^\mu \quad \text{and} \quad \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0.$$

To compute the conserved current, we must find $\delta\psi$. We know $\psi \to \psi' = e^{i\alpha\gamma^5} \sim (1 + i\alpha\gamma^5)\psi$, so $\delta\psi \sim i\gamma^5\psi$. Therefore, our conserved current is

$$\boxed{j_5^\mu = -\bar{\psi}\gamma^\mu\gamma^5\psi.} \tag{3.3}$$

Note that Peskin and Schroeder write the conserved current as $j_5^\mu = \bar{\psi}\gamma^\mu\gamma^5\psi$. This is essentially equivalent to the current above and is likewise conserved.

e) We are to compute the divergence of the Noether current generally (i.e. when there is a possibly non-zero mass). We note that the Dirac equation implies that $\gamma^\mu \partial_\mu \psi = -im\psi$ and $\partial_\mu \bar{\psi}\gamma^\mu = im\bar{\psi}$. Therefore, we may compute the divergence directly.

$$\begin{aligned}
\partial_\mu j_5^\mu &= -(\partial_\mu \bar{\psi})\gamma^\mu\gamma^5\psi - \bar{\psi}\gamma^\mu\gamma^5\partial_\mu\psi, \\
&= -(\partial_\mu \bar{\psi})\gamma^\mu\gamma^5\psi + \bar{\psi}\gamma^5\gamma^\mu\partial_\mu\psi, \\
&= -im\bar{\psi}\gamma^5\psi - im\bar{\psi}\gamma^5\psi,
\end{aligned}$$

$$\boxed{\therefore \partial_\mu j_5^\mu = -i2m\bar{\psi}\gamma^5\psi.} \tag{3.4}$$

Again, this is consistent with the sign convention we derived for j_5^μ but differs from Peskin and Schroeder.

4. a) We are to find unitary operators $\mathcal{C}$ and $\mathcal{P}$ and an anti-unitary operator $\mathcal{T}$ that give the standard transformations of the complex Klein-Gordon field.

Recall that the complex Klein-Gordon field may be written

$$\begin{aligned}
\phi(x) &= \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{\mathbf{p}}e^{-i\mathbf{px}} + b^\dagger_{\mathbf{p}}e^{i\mathbf{px}}\right); \\
\phi^*(x) &- \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a^\dagger_{\mathbf{p}}e^{i\mathbf{px}} + b_{\mathbf{p}}e^{-i\mathbf{px}}\right).
\end{aligned}$$

We will proceed by ansatz and propose each operator's transformation on the ladder operators and then verify the transformation properties of the field itself.

Parity

We must to define an operator $\mathcal{P}$ such that $\mathcal{P}\phi(t,\mathbf{x})\mathcal{P}^\dagger = \phi(t,-\mathbf{x})$. Let the parity transformations of the ladder operators to be given by

$$\mathcal{P}a_{\mathbf{p}}\mathcal{P}^\dagger = \eta_a a_{-\mathbf{p}} \qquad \text{and} \qquad \mathcal{P}b_{\mathbf{p}}\mathcal{P}^\dagger = \eta_b b_{-\mathbf{p}}.$$

We claim that the desired transformation will occur (with a condition on η). Clearly, these transformations imply that

$$\mathcal{P}\phi(t,\mathbf{x})\mathcal{P}^\dagger = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(\eta_a a_{-\mathbf{p}}e^{-i\mathbf{px}} + \eta_b^* b^\dagger_{-\mathbf{p}}e^{i\mathbf{px}}\right) \sim \phi(t,-\mathbf{x}).$$

If we want $\mathcal{P}\phi(t,\mathbf{x})\mathcal{P}^\dagger = \phi(t,-\mathbf{x})$ up to a phase η_a, then it is clear that η_a must equal η_b^* in general. More so, however, if we want true equality we demand that $\eta_a = \eta_b^* = 1$.

Charge Conjugation

We must to define an operator $\mathcal{C}$ such that $\mathcal{C}\phi(t,\mathbf{x})\mathcal{C}^\dagger = \phi^*(t,\mathbf{x})$. Let the charge conjugation transformations of the ladder operators be given by

$$\mathcal{C}a_{\mathbf{p}}\mathcal{C}^\dagger = b_{\mathbf{p}} \qquad \text{and} \qquad \mathcal{C}b_{\mathbf{p}}\mathcal{C}^\dagger = a_{\mathbf{p}}.$$

These transformations clearly show that

$$\mathcal{C}\phi(t,\mathbf{x})\mathcal{C}^\dagger = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(b_{\mathbf{p}}e^{-i\mathbf{px}} + a^\dagger_{\mathbf{p}}e^{i\mathbf{px}}\right) = \phi^*(t,\mathbf{x}).$$

Time Reversal

We must to define an operator $\mathcal{T}$ such that $\mathcal{T}\phi(t,\mathbf{x})\mathcal{T}^\dagger = \phi(-t,\mathbf{x})$. Let the anti-unitary time reversal transformations of the ladder operators be given by

$$\mathcal{T}a_{\mathbf{p}}\mathcal{T}^\dagger = a_{-\mathbf{p}} \qquad \text{and} \qquad \mathcal{T}b_{\mathbf{p}}\mathcal{T}^\dagger = b_{-\mathbf{p}}.$$

Note that when we act with $\mathcal{T}$ on the field ϕ, because it is anti-unitary, we must take the complex conjugate of each of the exponential terms as we 'bring $\mathcal{T}$ in.' This yields the transformation,

$$\mathcal{T}\phi(t,\mathbf{x})\mathcal{T}^\dagger = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\mathbf{p}}}}\left(a_{-\mathbf{p}}e^{i\mathbf{px}} + b^\dagger_{-\mathbf{p}}e^{-i\mathbf{px}}\right) = \phi(-t,\mathbf{x}).$$

b) We are to check the transformation properties of the current

$$J^\mu = i[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi],$$

under $\mathcal{C}$, $\mathcal{P}$, and $\mathcal{T}$. Let us do each in turn.

Parity

Note that under parity, $\partial^\mu \to \partial_\mu$.

$$\begin{aligned}\mathcal{P}J^\mu\mathcal{P}^\dagger &= \mathcal{P}i[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi]\mathcal{P}^\dagger,\\ &= i[\mathcal{P}\phi^*\mathcal{P}^\dagger\mathcal{P}\partial^\mu\phi\mathcal{P}^\dagger - \mathcal{P}\partial^\mu\phi^*\mathcal{P}^\dagger\mathcal{P}\phi\mathcal{P}^\dagger],\\ &= i[\phi^*(t,-\mathbf{x})(\partial_\mu\phi(t,-\mathbf{x})) - (\partial_\mu\phi^*(t,-\mathbf{x}))\phi(t,-\mathbf{x})],\end{aligned}$$

$$\boxed{\therefore \mathcal{P}J^\mu\mathcal{P}^\dagger = J_\mu.} \tag{4.1}$$

Charge Conjugation

$$\begin{aligned}\mathcal{C}J^\mu\mathcal{C}^\dagger &= \mathcal{C}i[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi]\mathcal{C}^\dagger,\\ &= i[\mathcal{C}\phi^*\mathcal{C}^\dagger\mathcal{C}\partial^\mu\phi\mathcal{C}^\dagger - \mathcal{C}\partial^\mu\phi^*\mathcal{C}^\dagger\mathcal{C}\phi\mathcal{C}^\dagger],\\ &= i[\phi(\partial^\mu\phi^*) - (\partial^\mu\phi)\phi^*],\end{aligned}$$

$$\boxed{\therefore \mathcal{C}J^\mu\mathcal{C}^\dagger = -J^\mu.} \tag{4.2}$$

Time Reversal

Note that under time reversal, $\partial^\mu \to -\partial_\mu$ and that $\mathcal{T}$ is anti-unitary.

$$\begin{aligned}\mathcal{T}J^\mu\mathcal{T}^\dagger &= \mathcal{T}i[\phi^*(\partial^\mu\phi) - (\partial^\mu\phi^*)\phi]\mathcal{T}^\dagger,\\ &= -i[\mathcal{T}\phi^*\mathcal{T}^\dagger\mathcal{T}\partial^\mu\phi\mathcal{T}^\dagger - \mathcal{T}\partial^\mu\phi^*\mathcal{T}^\dagger\mathcal{T}\phi\mathcal{T}^\dagger],\\ &= -i[-\mathcal{T}\phi^*\mathcal{T}^\dagger(\partial_\mu\mathcal{T}\phi\mathcal{T}^\dagger) + (\partial_\mu\mathcal{T}\phi^*\mathcal{T}^\dagger)\mathcal{T}\phi\mathcal{T}^\dagger],\\ &= i[\phi^*(-t,\mathbf{x})(\partial_\mu\phi(-t,\mathbf{x})) - (\partial_\mu\phi^*(-t,\mathbf{x}))\phi(-t,\mathbf{x})],\end{aligned}$$

$$\boxed{\therefore \mathcal{T}J^\mu\mathcal{T}^\dagger = J_\mu.} \tag{4.3}$$